MATH 8820

Homework Assignment 3

Problem 1: Exploring other prior specifications. Based on conversations in class, I have decided to have you explore several alternate prior specifications for the multiple linear regression model; i.e., the model given by

$$Y_i = \mathbf{x}_i'\boldsymbol{\beta} + \epsilon_i,$$

where $\mathbf{x}_i = (1, x_{i1}, ..., x_{ip})'$, $\boldsymbol{\beta} = (\beta_0, \beta_1, ..., \beta_p)'$, and $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$.

(a) Find the paper entitled "Power Prior Distributions for Regression Models," by Ibrahim, J. and Chen, M. Provide a brief discussion of the idea of how a power prior is used to incorporate information obtained from historical data. After doing this, derive the necessary details to implement this idea for the model discussed above. Be sure to differentiate between historical and current data in your notation.

(b) Zellner A. (1986) proposed what is referred to as a G-prior, find a reference and briefly describe what a G-prior is and how it is developed. After doing this, derive the necessary details to implement this idea for the model discussed above. Note, the zlm function in the BMS package in R makes use of this approach.

(c) Write a general R function for fitting the model above and provide detailed instructions on how one could use this function to fit the model under the priors discussed in (a) and (b).

(d) Write a small simulation study to compare these two prior choices to one another, and also compare them to the multiple linear regression model discussed in class. Note, to implement the power prior you will have to simulate a historical data set, do so wisely.

Problem 2: Consider the spatial regression model which is given by

$$Y_i = \beta_0 + b_i + \epsilon_i,$$

where $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$, for $i = 1, ..., n$. To capture spatial dependence we will assume that the spatial random effects follow a conditional autoregressive (CAR) model; i.e.,

$$\mathbf{b} = (b_1, ..., b_n)' \sim CAR(\tau^2, \rho),$$

or equivalently

$$\mathbf{b} = (b_1, ..., b_n)' \sim N\{\mathbf{0}, \tau^2(\mathbf{D} - \rho\mathbf{W})\},$$

where $\mathbf{D}$ is a diagonal matrix whose ith diagonal element denotes the number of neighbors the ith spatial unit possess, and $\mathbf{W}$ is the usual neighborhood matrix; i.e., $\mathbf{W}_{ik} = 1$ if the ith spatial unit is a neighbor of the kth spatial unit, and $\mathbf{W}_{ik} = 0$ otherwise. Note $\mathbf{D}_{ii} = \sum_{k=1}^{n} \mathbf{W}_{ik}$ and $\mathbf{W}_{ii} = 0$, for all i. For the purposes of this problem we will elicit the following priors:

$$\begin{aligned} \pi(\beta_0) &\propto 1, \\ \sigma^{-2} &\sim \text{Gamma}(a_\sigma, b_\sigma), \\ \tau^{-2} &\sim \text{Gamma}(a_\tau, b_\tau). \end{aligned}$$

For the purposes of this exercise we will assume ρ is fixed. Based on this model complete the following:

(a) Based on the CAR model, derive the conditional distribution of $b_i|\mathbf{b}_{(-i)}$, where $\mathbf{b}_{(-i)} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)'$. Note, this derivation does not make use of the observed data.

(b) Find the following full conditional distributions; $\beta_0|$else, $\sigma^{-2}|$else, $\tau^{-2}|$else, $\mathbf{b}|$else, and $b_i|$else. In your final expression for each explicitly state the dependence.

(c) Here you will develop two Gibbs sampling algorithms which can be used to fit the aforementioned model; in the first you will update all of the spatial random effects simultaneously and the other will update the spatial random effects one at a time. In doing so you will write out a symbolic sampling algorithm which you will hand in as well as writing separate functions in R which can be used to fit the model. Coding efficiency is important in this exercise, so pay attention to how you put things together.

(d) To examine how your code works, you will then use both of the aforementioned R functions to fit the spatial model to the data created by the following code.

```
library(lattice)  # You may have to download this package
P1<-50
P2<-50
Y<-matrix(-99,P1,P2)
for(p1 in 1:P1){
for(p2 in 1:P2){
Y[p1,p2]<- 10*dnorm((p1-p2),0,10)+ rnorm(1,0,0.1)
}}
levelplot(Y)
```

Note, in this application the individual pixels in the image will be viewed as the spatial units, and the observed data is stored in the matrix given by Y. To fit the model you will have to label each spatial unit in this image and then construct the necessary quantities based on your labeling. Once this is complete, fit the spatial model using both of the R functions you developed in part (c). Discuss the results of your analysis, further provide figures that summarize your estimates of the **b** (e.g., mean, standard deviation, etc.), these figures should constructed to correspond with the data that you analyze. Note, to complete all of this you will have to specify ρ, do so in a reasonable way and provide a brief discussion.